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Exam II , MTH 221 , Fall 2010, 2pm section

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QUESTION 1. **(12 points)** Let $F = span\{1 - x + x^2, -1 + x^2, -x + 2x^2\}$

(i) Find a basis for F.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for F: $\{1 - x + x^2, -1 + x^2\}$

(ii) Is $1 - 3x + 5x^2 \in F$? EXPLAIN. If YES, write $1 - 3x + 5x^2$ as a linear combination of the basis elements.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 1 & -3 & 5 \end{bmatrix} \xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Yes, it belongs to F as it can be [illegible]

QUESTION 2. **(5 points)** Given $v_1 = (2, 3, 2), v_2 = (-2, -3, 2)$ are independent in R^3. Find $v_3 \in R^3$ such that $B = \{v_1, v_2, v_3\}$ is a basis for R^3. Show the work.

$$\begin{bmatrix} 2 & 3 & 2 \\ -2 & -3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2+R_3 \to R_3} \begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

$v_3 = (0, 1, 1)$

QUESTION 3. **(5 points)** Are $2 + x + 7x^2, -2 + 13x - 5x^2, 4 + x^2, -7x + 13x^2$ independent in P_3?explain (you may finish on the back)

$$\begin{bmatrix} 2 & 1 & 7 \\ -2 & 13 & -5 \\ 4 & 0 & 1 \\ 0 & -7 & 13 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 14 & 2 \\ 0 & -2 & -13 \\ 0 & -7 & 13 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2+R_4 \to R_4]{\frac{1}{7}R_2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 14 & 2 \\ 0 & 0 & -\frac{89}{7} \\ 0 & 0 & 14 \end{bmatrix}$$

(ii) $1-3x+5x^2 = \alpha_1(1-x+x^2)+\alpha_2(-1+x^2)$

$(1,-3,5) = \alpha_1(1,-1,1)+\alpha_2(-1,0,1)$

$$\left.\begin{aligned} 1 &= \alpha_1 - \alpha_2 \\ -3 &= -\alpha_1 \\ 5 &= \alpha_1 + \alpha_2 \end{aligned}\right| \quad \begin{aligned} \alpha_1 &= 3 \\ \alpha_2 &= 2 \end{aligned}$$

$\therefore$

$1-3x+5x^2 = 3(1-x+x^2)+2(-1+x^2)$

Q.3

$$\xrightarrow{\frac{98}{89}R_3+R_4 \to R_4} \begin{pmatrix} 2 & 1 & 7 \\ 0 & 14 & 2 \\ 0 & 0 & -89/7 \\ 0 & 0 & 14 \end{pmatrix}$$

$x(-89/7)+14=0$

$x = \dfrac{-14\times 7}{-89}$

$x = 98/89$

Not independent

as $-7x+13x^2$ can

be written as a

linear combination

of the others

QUESTION 4. (15 points) Given $T : P_4 \longrightarrow R_{2\times 2}$ such that $T(f(x)) = \begin{bmatrix} f(-1) & f(0) \\ f(-1) & f(0) \end{bmatrix}$ is a linear transformation.

(i) Find the standard matrix representation of T.

$(1, x, x^2, x^3)$

$T(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad T(x^3) = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$

$T(x) = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(ii) Find a basis for Ker(T) and write Ker(T) as a span.

$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_1+R_4 \to R_4]{-R_1+R_2 \to R_2,\ -R_1+R_3 \to R_3} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[-R_2+R_4 \to R_4]{R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & -1 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{bmatrix}$

free $= x_3, x_4$

$x_1 = 0$

$x_2 - x_3 + x_4 = 0$

$N(A) = \{(0, x_3 - x_4, x_3, x_4)\}$

$x_3 = 1, x_4 = 0$

basis for Ker(T) $= \{x + x^2, -x + x^3\}$

$(0, 1, 1, 0)$ $\quad (0, -1, 0, 1)$ $\quad$ Ker(T) = span$\{x + x^2, -x + x^3\}$

$x_3 = 0, x_4 = 1$

(iii) Find a basis for Range(T) and write Range(T) as a span.

col(A) $= \{(1,1,1,1), (-1,0,-1,0)\}$

basis Range(T) $= \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$

Range(T) = span$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \right\}$

QUESTION 5. **(15points)** Given $T : P_3 \longrightarrow R$ is a linear transformation such that $T(2) = 6$, $T(x^2 + x) = -5$, and $T(x^2 + 2x + 1) = 4$.

(i) Find $T(x)$ and $T(x^2)$ and $T(5x^2 + 3x + 8)$

$T(x^2) + T(x) = -5$

$T(x^2) + 2T(x) + T(1) = 4$

$T(x^2) = -5 - T(x)$

$-5 - T(x) + 2T(x) + T(1) = 4$

$T(x) + \frac{1}{2}T(2) = 9$

$T(x) + 3 = 9$

$T(x) = 6$

$T(x^2) = -5 - 6$

$T(x^2) = -11$

$\Rightarrow$ next page

(ii) Find the standard matrix representation for T.

$T(1) = 3$

$T(x) = 6$

$T(x^2) = -11$

$A = \begin{bmatrix} 3 & 6 & -11 \end{bmatrix}$

(iii) Find a basis for Ker(T) and write Ker(T) as a span.

$\left[\begin{array}{ccc|c} 3 & 6 & -11 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -11/3 & 0 \end{array}\right]$

$x_1 + 2x_2 - \frac{11}{3}x_3 = 0$

$x_1 = -2x_2 + \frac{11}{3}x_3$

$N(A) = \{ -2x_2 + \frac{11}{3}x_3, x_2, x_3 \}$

$x_2 = 1, x_3 = 0$

$(-2, 1, 0)$

$x_2 = 0, x_3 = 1$

$(11/3, 0, 1)$

$\Rightarrow$ next page

(iv) Is T ONTO? explain.

QUESTION 6. **(15 points)** Let $F = \{(a, b, c, d) \in R^4 \mid a, b, c, d \in R, a + 2b + 3d = 0, \text{ and } c - 3b + d = 0\}$.

(i) Show that F is a subspace of R^4.

$a = (1)a + 0(b) + 0(c) + 0(d)$

$b = 0(a) + 1(b) + 0(c) + 0(d)$

$c = 0(a) + 0(b) + 1(c) + 0(d)$

$d = 0(a) + 0(b) + 0(c) + 1(d)$

as a, b, c, d can be written as a linear combination of a, b, c, d, F is a subspace of R^4

(ii) Find a basis for F and write F as a span.

$a = -2b + 3d$, $c = 3b + d$

$F = \{(-2b + 3d, b, 3b + d, d) \in R^4 \mid b, d \in R\}$

$b = 1, d = 0$

$(-2, 1, 3, 0)$

$b = 0, d = 1$

$(3, 0, 1, 1)$

basis for $F = \{(-2, 1, 3, 0), (3, 0, 1, 1)\}$

$F = \text{span}\{(-2, 1, 3, 0), (3, 0, 1, 1)\}$

$$T(5x^2+3x+8) = 5\,T(x^2) + 3\,T(x) + 4\,T(2)$$

$$= 5 \times (-11) + 3(6) + 4(6)$$

$$= -13$$

(iii)

basis $\ker(T) = \{-2+x,\ 11/3 + x^2\}$

$\ker(T) = \operatorname{span}\{-2+x,\ 11/3 + x^2\}$

basis $\operatorname{Rang}(T) = \{3\} \implies$ dim of $\operatorname{Rang}(T) = 1 = \dim(\mathbb{R}$

$\therefore$ T is onto

QUESTION 7. (8 points) Let $D = \{3a + (2+b)x^2 + 4ax^3 \mid a, b \in R\}$ Is D a subspace of P_4? If NO, explain. If YES, find a basis for D

No, D is not a subspace since 2+b cannot be written as a linear combination of a and b.

QUESTION 8. (15 points) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 5 & 2 \end{bmatrix}$

(i) Find a basis for $ROW(A)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 5 & 2 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

Basis Row(A) = $\{(1,1,1,1,1,1), (0,2,0,3,4,5), (0,0,0,0,3,0)\}$

(ii) Find a basis for $Col(A)$

basis Col(A) = $\{(1,-1,2), (1,1,2), (1,3,5)\}$

QUESTION 9. (10 points) Given $L = \{\begin{bmatrix} 6a & 2a+3b \\ 2b & -c \end{bmatrix} \mid a, b, c \in R\}$ is a subspace of $R_{2\times 2}$. Find a basis for L.

a = 1, b = 0, c = 0

$\begin{bmatrix} 6 & 2 \\ 0 & 0 \end{bmatrix}$

a = 0, b = 1, c = 0

$\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$

a = 0, b = 0, c = 1

$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

basis for L : $\left\{ \begin{bmatrix} 6 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

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